Is a continuum denumerable?[1]

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From the set-theoretical point of view, the question forming the title of this paper seems very strange, almost nonsensical, because – as we are usually told – Cantor *proved* that the totality of real numbers cannot be denumerable and the problem said to be facing us is merely what non-denumerable cardinality it has.

What I firstly wish to emphasize is that there is a significant difference between the diagonal construction, as such, and the diagonal argument itself, which is susceptible to varying interpretations. The most common interpretation shows (1) that there are *more* reals than integers, and hence (2) that some of the reals must be independent of language, because the totality of words and sentences is always countable. The majority of mathematicians take this argument and its "consequences" for granted, not realizing that they have trespassed over the boundary into the wide and insecure field of philosophy. And their very belief that they are merely articulating an obvious, mathematical fact worsens the situation for the following reason:

In the long, honorable tradition of Plato, Kant and Wittgenstein, mathematics, due to its complicated, indirect relation to the empirical world, has always played an important heuristic role in exposing the delicacy of the relationship between our so-called reality (of scientific facts) and its description, and the rashness of simply presupposing that there lies between them some precise, pre-given correspondence that we can only reveal but not affect. So, making the same assumption in the realm of mathematics both fails to help us explain what its sentences are about and deprecates this whole tradition as well.

In view of this, my intention here is to repeat what I have previously[2] attempted for the intermediate value theorem of Bolzano and the recursion theorem of Dedekind, namely to show that we are not dealing with proofs in the sense of arbitrating between the truth or falseness of a given conjecture, but rather that we are dealing with something more like resolutions to proceed in some specific way, which is only one among many ways not delimited in advance. Such deci-

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[2] See Kolman (2005), Kolman (2007).

sions must underlie real proofs (of the real facts) to be articulated later and I will argue that there are "ontologically" safer ways of developing the diagonal argument into a full-fledged theory of continuum than Cantor's own proposal. By corollary, resolutions to other famous semantic paradoxes based on diagonal construction can then be established as well.

1

To start off, let me put my question into its historical perspective. L.E.J. Brouwer is rightly credited for his violent and yet brave attack on the contemporary foundations of mathematics, including many allegedly unquestionable "scientific facts" like *tertium non datur*. It was his accent on the conventional nature of these "facts" or, in other words, the possibility of doing the old things differently, which famously made an impact on Wittgenstein. However, just like Napoleon before him, also Brouwer, after some time of promising a real revolution, proclaimed himself an emperor, replacing the old, "obvious" facts with new, "intuitive" ones. This unpleasant switch, I claim, was the result of his unwillingness to take seriously the linguistic turn, initiated by Frege and elaborated by Wittgenstein.

Interestingly, Brouwer's first failure recalls the dilemma of the French semi-intuitionists who, whilst managing to convince themselves that a continuum, like everything in the world (of mathematics), must be language-dependent and so denumerable, nevertheless strived desperately to keep it non-denumerable in order to safeguard its convenient topological and metrical properties. Having replaced language as the medium of all knowledge by "primordial intuition", Brouwer not only did not resolve the dilemma, but actually deepened it by introducing the separate concepts of *practical, denumerable* continuum as opposed to the *full, non-denumerable* one which in his opinion is independent of language. As a result, the constructive twist of his original approach was completely lost.

Hilbert, on the other hand, despite his fanatical and "politically" motivated finitist reaction to Brouwer's "putsch" and Weyl's "betrayal", can be seen as an initiator of the true switch of paradigms that led through his original axiomatic method to the thesis that all our knowledge rests on inferential grounds. In Hilbert's writings, remarkably, this argument takes the form of a transcendental deduction *sui generis*, starting with the words: in the beginning was a sign.

2

Against this (say) holistic background it seems inevitable that even in set theory or analysis one is always committed to starting with seemingly obvious questions like “what is a continuum?” and “what does it take for it to be bigger than the set of natural numbers?”. There is then no way of answering them (in a language) without making the real numbers or sets language-dependent, and this amounts to describing

(1) their language representations (e.g. $\lim(a_n)_{n\in N}$ for $(a_n)_{n\in N}$ Cauchy sequence of rational numbers) and
(2) the relevant context of predicates (e.g. $x + y = z$, $x < y$ etc.) in which these representations can be intersubstituted *salva veritate* with respect to
(3) the identity criteria chosen in advance (e.g. $\lim(a_n)_{n\in N} = \lim(b_n)_{n\in N}$ iff $\lim(a_n-b_n)_{n\in N} = 0$, where the “lim” on the right side stands for the rational convergence.)

In the technical sense, the identity criteria are the vehicle responsible for making a name a name of something, i.e. for the enterprise of reference.

As since Weierstrass we have become accustomed to grasping the reals as the sequences of natural or rational numbers, it follows that one must first explain the concept of function. Interestingly, our second problem, how many reals there are, returns us to the same basic question, because of Cantor’s decision to compare the sizes of two sets in terms of their one-to-one correspondence.

None of these decisions is “natural” or somehow “in the air”. As is commonly known, Cantor’s idea was considered counterintuitive at first, at variance with Euclid’s “common notion” that the whole is greater than the parts. The continuum, on the other hand, has a very intricate conceptual development beginning with the Pythagorean definition of proportion by means of a reciprocal subtraction (*anthyphairesis*) and the Euclidian theory of points constructible by means of a ruler and a compass. In modern times, these were superseded by the Cartesian idea of numbers as roots of polynomials, and today we tend to work within the theory of *lawlike* or *lawless* sequences of rational numbers.

Now, if you have some simple system of naming functions (i.e. some names and identities between them), as, e.g., that of the primitive recursive arithmetic, you can make an exhaustive list of their names $f_1, f_2, f_3, \ldots$, and form the name $\mathrm{diag}(x) = f_x(x) + 1$ of a function which, by definition, is not there. So, to every *schematically given*, and thereby enumerable totality of functions there is a function absent from the list.

But this is all we have; in particular, we are not entitled to say that the totality of all functions is non-denumerable because (1) “all functions” means “all extensions of the names in question”, hence, according to our *internal* definition, diag is not a function, at least not of the same type. And (2) if for some *external* reason we are willing to call it a function, then the argument shows (not proves) something to the effect that the expression diag transcends the expressive possibilities of the initial system and, by implication, that the original definition of function is too narrow. Hence, the most charitable reading of Cantor's construction renders it as a pre-theoretical justification for delimiting what constitutes a continuum more liberally.

In recursion theory, this observation led to the concept of general or partial recursive function. Cantor, however, thought "one-dimensionally", as metaphysicians do, without taking the enterprise of a reflective metalanguage into account. This made him start from what should have been the ultimate result of the defining process, leaving its intrinsic relativity or language-dependency aside. By grasping the function as the most liberal extension of the concept of right-unique correspondence, he believed himself to have won the whole game by "fiat". Later, he adopted the same strategy towards the concept of set, describing it as the most general product of collecting objects together, either with the help of some property or by means of some kind of generalized counting. In this case, however, the shortcomings of his method came to light quite quickly, with the entry of the paradox: the only response Cantor could offer was to call problematic sets inconsistent, which, as happens with every Platonist theory, amounts to saying that there is what there is and there is not what there is not. But our search is not for such tautologies, but rather for non-trivial criteria telling us which possible names count as names of something. And this is exactly to say that we must start from language.

Those who still feel some sympathy with Cantor's way of thinking should realize that just as he defined a real number as an *arbitrary* (Cauchy) sequence, it would have been possible for the Greeks to define a real number as a point constructible by whatever means, or for Gödel to say that provable is everything that is true. By that, however, famous problems like the quadrature of the circle, the axiomatizability of arithmetic or the "Enscheidungsproblem" would have simply vanished: Archimedes was able to draw a line corresponding to the circumference of a given circle; and if I give you a formula, I am quite sure you will be able to decide, in a finite number of steps, whether it is a tautology or not. What you might not be able to do, however, is to solve the problem with pre-chosen canonical methods such as by means of one particular Turing machine.

So, it turns out that the best means we have at our disposal at the given moment (ruler and compass, deductive proof-system, Turing machine etc.) actually matters for deciding what there is and what there is not. And because these methods are the best available *at the time*, they are also limited and, hence, replaceable. But this actually follows already from the simple fact that explanation must be in some sense simpler than what is to be explained by it.

As for the diagonal argument, the conclusion already drawn by Wittgenstein is straightforward: it is inappropriate and misleading to say there are more reals than integers, the only fact being that their names are used differently. This observation caused Brouwer, at first, to call his continuum *denumerably unfinished* in the sense that it cannot form a closed, schematically given collection in pain of a paradox. It is not static, but freely developable in many possible directions, depending on pre-defined stages. Since each of these stages is obviously denumerable, so is the whole system, but this time in another, unfinished sense. Abandoned by Brouwer, this conception of continuum was later elaborated by Lorenzen in his operative mathematics. Its main advantage is that it does not make a continuum independent of language in general, but only of languages delimited too narrowly, as, e.g., in Frege's *Grundgesetze* or within the recursion theory.

3

But interestingly, even in the case of these limited languages it is possible to introduce a sound concept of continuum that accords with the salient part of Cantor's argument. One only has to take advantage of the fact that even a schematically given infinity does not have to be effectively controllable. This coincides with the original insight of Brouwer, on which he based his attack on the principles of classical logic: even if you limit yourself to the sequences governed by finite laws, you are generally not able to decide effectively whether they name the same function or not, because their courses-of-values are essentially infinite. Hence, if interpreted effectively, the principle of the excluded middle fails to hold.

Brouwer himself did not take full advantage of this possibility, devising instead a continuum of his own governed by the principle of "free choice". His free choice sequences, such as that generated by dice throwing, are by definition not only incontrollable effectively, but not controllable at all. As a result, this move draws Brouwer's basic attitude closer to that of Cantor's than either of the two mathematicians would have been willing to admit, implying as it does that intuitionism and Platonism may after all be two sides of the same coin differing only on a verbal level.

By contrast, the starting point of the so-called recursive analysis is more modest and powerful, as far as the plausibility and transparence of the results is concerned. The totality of all partial recursive functions is schematically given to the extent that it can be produced by a machine in the way in which all integers can. This practical enumerability is identified with the so-called recursive enumerability or enumerability by way of a recursive function which is every partial recursive function that is total.

Now, due to their schematic or syntactic characterization you can easily make a list of all partial recursive functions. In this enumeration, all recursive functions are contained by definition, building enumerable totality in Cantor's sense, whereby a subset of an enumerable set is enumerable. But in our algorithmic, computable sense this enumerability is only fictitious, because there is no effective way of separating the non-total functions from the total ones, and hence, recursive functions are (recursively) non-denumerable.

The decisive point of the whole turn lies in the fact that the ordering of cardinalities is defined negatively as the non-existence of a certain function:

> Card(A)<Card(B) iff there exists a one-to-one function from A to B, but *not* one from A onto B.

Obviously, we have something like a positive version of the Löwenheim-Skolem paradox here, which is no paradox once we realize, again, that there is no concept of function independent of us.

Along these lines, within the so-called recursive analysis, one can mimic many of Cantor's and Brouwer's results on safe non-ontological grounds. The (recursive) non-denumerability of the continuum has this time obviously nothing to do with its "size", which is clearly denumerable in Cantor's sense, and, as a result, we cannot repeat his argument to get bigger and bigger cardinalities. So, we are not leaving the grounds of the nameable entities at all. And if you are still prone to think about it as an underhand trick, just remember that recursive delimitation of a function does have plausibility as it stems from its primitive meaning of an algorithmic procedure.

As for the other results on Brouwer's side, we have some believable counterparts to his inventive, though slightly insane proofs such as that "every total real function is (locally) uniformly continuous", or that

> a continuum is indecomposable, i.e. cannot be (effectively) divided into two non-empty parts.

The recursive alternative to this claim is one of the basic results in recursion theory, the so-called Rice's theorem, announcing that

> any nonempty proper subset of the set of all partial recursive functions (i.e. the union of all equivalence classes of their names) is not decidable.

Classically, this result corresponds to the claim that

> a continuum is connected, i.e. not a disjoint union of two non-empty open sets,

which is usually considered a theoretical approximation of the famous Aristotelian description of the continuity as the coincidence of boundaries of two things that touch each other. In view of the variety of properties which we have at our disposal today – such as "density", "connectedness", "compactness", "completeness (wrt an order or a metric)" etc. – this makes no sense. It only shows, once again, that continuum or continuity is not something immediately given to us, either in intuition or in pure practice (as the followers of Wittgenstein would prefer to say), but comes to incorporate a mixture of various theoretical and practical purposes and limitations, not predictable in advance.

4

For my conclusion, let me apply some of the observations made above about real numbers within the broader area of the philosophy of language. The topic I want to address is that comprised by the so-called semantic paradoxes, in particular those related to Cantor's diagonal argument. I shall restrict myself to a simpler one, associated with the name of Jules Richard, which goes like this:

> Consider the set E of all finitely definable reals and form a name diag by applying Cantor's diagonal construction to the members of E; then the number named by diag should be in E, because diag is finite, but at the same time it should not be there because of the way it was constructed.

What I claim now is that the nub of these antinomies does not lie in their auto-referential structure, but in their handling of some tacit presuppositions to do with naming numbers or naming in general. Let me first split the argument into the following four steps:

1. We have a language over a finite alphabet, e.g., that of all of the characters producible by a certain typewriter.
2. We fix some "alphabetical" order of these signs and make an enumeration of all their possible sequences.
3. We go through this list and delete all expressions that do not define or characterize a real number.
4. Eventually we get an enumeration $(d_n)_{n \in N}$ of all names of real numbers available in the given language, and consequently an enumeration $(\mathbf{d_n})_{n \in N}$ of all real numbers definable in this way, $\mathbf{d_n}$ being the refernce of dn. But the name diag:="the diagonal number of $(\mathbf{d_n})_{n \in N}$" does not, by definition, denote any of the numbers $(\mathbf{d_n})_{n \in N}$, and hence it cannot be in $(d_n)_{n \in N}$; where it, on the other hand, belongs.

Although points 1 and 2 are harmless and perfectly acceptable in the case of any simple or artificial language, the paradox's strength comes from its application to natural language. First, there are many natural languages with many different alphabets, none of them being *the* most basic or most natural, and, second, you can enlarge such a language freely and indefinitely, e.g. by signs like @, $, # etc. Incidentally, I consider it no accident that what Zermelo called Skolemism, and what he actually made responsible for the paradox, was precisely this idea of representing every concept by means of a fixed set of signs. But let us assume, for the sake of argument, that this does not matter, that our typewriter is good enough for us, and let us turn now to point 3.

It is obvious that going through the list and deleting all the inappropriate expressions will sooner or later lead us to encounter the expression diag="the diagonal number of $(d_n)_{n\in N}$". The paradox arises because diag looks like the name of a number defined in terms of a reference to the sequence $(d_n)_{n\in N}$. And that is also true once the sequence $(d_n)_{n\in N}$ is formed, i.e. after all the inappropriate signs are deleted. The point is that, just like an expression such as "this cat", diag depends on its being a genuine name in the context of utterance, on the presupposition that there is some sequence (or a cat) – within the (logical) space we are situated in.[3] Since in the course of creating the sequence $(d_n)_{n\in N}$ this sequence does not yet exist, the expression diag is not a name and must be erased, which solves the paradox.

In this reading, the paradox results from our conceiving names and other meaningful symbols in an oversimplified way, as constituted by some simple syntactic criteria. But *name* is not a syntactic, but a semantic category; it is a name of something, and what the diagonal argument shows is precisely that the existence of an appropriate reference relation cannot be generally resolved by means of merely surface grammatical criteria.

5

There is, of course, a striking similarity between this argument and the argument for the (recursive) non-denumerability of recursive functions. The difference is that this time, instead of doubting our (finite) faculty of recognizing an expression as being the name of something (infinite in nature), we are doubting the independence of reference of the actual context, constituted by what has already been defined and what has not yet been defined. We have acclimatized ourselves fairly well to this indexical quality of words in our everyday world, but not as yet in the world of mathematics, which is usually taken to be eternal, i.e. totally time- and space-independent.

In contrast to this attractive, yet superficial construal, we can have a quite plausible theory of the continuum which depends in its form always on what has been to a considerable extent quite deliberately defined so far, i.e. up to this moment. To avoid misunderstanding: what I am claiming now should be

[3] This observation is due to Stekeler-Weithofer (1986).

nothing controversial, of the kind that there is an intuition of time behind all of mathematics, or that we must incorporate some dynamic into it. Partly because of this, and also for the sake of contrast, I have dealt with the conceptually static methods of recursive analysis.

Anyway, my point is that within the transient world of our experience nothing is absolutely stable, i.e. every kind of stability is always relative to our faculties of recognizing the different things as the same in some sense. What begs the question here, of course, is in which sense the sentences of mathematics are *more* stable than other sentences. But the old-fashioned answer works here quite well: mathematics is a product of our reason, and hence it is under its exclusive control. My present objective, however, was the conventional character of all knowledge we have, i.e. without exception.

As for the title question: "is a continuum denumerable?", my answer is that it certainly could be. There is no "must" here. I have also nothing against Cantor's argument or against bigger cardinalities. I only do not take them for necessities or facts of reason, arguing that there are no such things in an absolute sense of the word. Of course, there are relative or transcendental necessities; like mathematical truths are to physics according to Kant, or the truths of logic are to mathematics according to Frege. In its rudimentary form of point set topology, Cantor's set theory could be thought of as the a priori of Weierstrassian analysis. As a general study of sets it became a kind of fiction, especially because the crucial concept of set was left unexplained, i.e. we do not know what the linguistic representations are that one is to quantify over. As a result, set theory, unlike arithmetic and (recursive) analysis, is basically the study of axiomatic systems, with no underlying pre-axiomatic concept of truth. Hence, the problems relating to the continuum hypothesis may be not only undecided at the moment (as, e.g., Goldbach's conjecture), but not decidable at all. We have simply not yet invested enough in the whole thing to get everything we want back.

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