Logicism and the Recursion Theorem[1]

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Old questions such as "Is logicism dead?" or "Are the truths of arithmetic synthetic a priori?" are once again being revived by scholars like Crispin Wright or George Boolos. But we have not learnt to be more cautious and still tend to answer them rashly without taking into account the broader background against which they were originally posed. As a consequence we are facing the same insurmountable difficulties as Poincaré, Wittgenstein or Russell, no matter whether their (rash) answers were "yes" or "no".

This paper has two aims. The first is to portray Frege's logicism in the spirit of Lakatos' logic of mathematical discovery as a bold conjecture eventually rejected. The second aim consists in showing that this rejection was based on different and more serious reasons than we are usually told. What I maintain is that one can agree with the neologicists that Frege's system is not so badly affected by Russell's paradox as was once thought, but still claim that this is not enough to render the whole project successful according to Frege's own standards. So the second aim of my paper is to indicate why – contrary to the neologicists' plan – the logicist idea cannot be saved *and* rendered compatible with Frege's original intentions, which, in my opinion, are quite sound. In view of this, of course, it is necessary to outline to some extent the original intentions and standards of Frege.[2] The recursion theorem and its role in the early foundational development turn out to be central to both parts of my argument.

1

The so-called recursion theorem (RT), first proved by Dedekind and later by Frege, pertains to the method of how functions on natural numbers can be uniquely defined, namely in the *usual* recursive way by (1) setting the value for 0 and (2) laying down the rules for computing the value at n+1 from the value at n. The theorem says that there is exactly one, i.e. one and only one, function

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[2] For a detailed account see Kolman (2005).

of this kind, thus guaranteeing the correctness and meaningfulness of the definition by recursion. Just for the record, the exact wording of the theorem (in one of its versions) goes like this:

> If *a* is an element of a set *S* and *g* is a function from *S* to itself, there is exactly one function *f* from the natural numbers to *S* such that (1) $f(0) = a$, (2) $f(n+1) = g(f(n))$ for every natural number *n*.

From the usual or "intuitive" point of view, however, there is no need, and in fact no room for such a license, because recursion is "intuitively" held for the most simple and natural way of proving or defining in arithmetic.

This intuitive point of view, moreover, seems to be supported by the fact that, for a long time, the theorem itself was omitted from books on foundations even by mathematicians such as Peano or Landau who, furthermore, did not explicitly share the idea of induction being the essence of arithmetical reasoning (in other words: who did not adhere to the Kantian philosophy of mathematics, unlike, e.g. Poincaré). Of course, these mathematicians may simply have missed the need for the theorem, as Landau actually thought he had when he – in his own words – added the proof of it to his *Foundations of Analysis* only after an intervention from a colleague.[3] As a result, it is possible to argue that it was only because of Frege's and Dedekind's foundational work and their proofs of certain seemingly "intuitive" theorems that the standards of rigor and the techniques of proof were gradually improved and mathematicians finally became aware of their necessity, as happened ostentatiously to Landau.

One can, furthermore, link Dedekind's proof with the general distrust of arguments based on intuition. Among the 19th century mathematicians this became a relatively common attitude to Kant's philosophy of mathematics with its thesis that the roots of arithmetic and geometry ought to be sought in the spatio-temporal structures imposed on reality by reason. The official standpoint, then, was that instead of trying to (mentally) intuit something we should (verbally) prove it, as Bolzano allegedly did in the case of the intermediate value theorem (IVT) or actually in its special case, the so-called Bolzano theorem which is, however, equivalent to the first one:

> Let, for two reals *a* and *b*, $a<b$, a function *f* be continuous on a closed interval $[a,b]$ such that $f(a)$ and $f(b)$ are of opposite signs. Then there exists a number $x \in [a,b]$ with $f(x)=0$.

Let us consider the tempting standard parallel between Bolzano's and Dedekind's theorems:

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[3] This observation is due to Michael Potter, see (Potter, 2000, p. 83).

We are usually told that Bolzano proved something self-evident on an analytic (purely conceptual or verbal) basis merely to avoid the spatial intuition behind it. Along these lines we can conclude that Frege and Dedekind tried to expel the temporal intuition from elementary arithmetic by replacing the recursive definition with sophisticated expressive and deductive tools. This actually does make sense because, according to Kant, the sequence of natural numbers is a result of counting in time, and this counting or construction in time is based on the routine recursive procedure: (1) produce the numeral 0, (2) provided n has already been constructed produce the numeral n + 1. Despite the fact that Frege and Dedekind themselves would have happily approved of this parallel, I suggest discarding it as misleading and replacing it with another, less obvious but certainly deeper as far as its consequences go.

What I want to draw your attention to is that both Bolzano's and Dedekind's proofs are in fact not proofs in the usual sense whereby a proof amounts to the "choice" between two basic possibilities, namely a conjecture's being true or false, as the proof of Fermat's theorem is or the proof of Goldbach's conjecture one day will be. The situation with the IVT and RT is different. Similarly to many other famous "proofs", such as Cantor's argument for the nondenumerability of the reals or Brouwer's proof that every total real function is (locally) uniformly continuous, what such "proofs" consist in is rather something like a resolution to proceed in a very specific way, which is only one among many ways not delimited in advance. Such resolutions (or, as Germans would say, prototheoretical justifications) can then underlie real proofs to be articulated later.

To illustrate the point, let us take a closer look at the IVT. There is no doubt that Bolzano's notorious example of an everywhere continuous but nowhere differentiable function can be read as the indication of certain troubles to which the explicit (not the intuitive) definitions of conceptualized analysis (such as those of continuity, convergence and derivative) can lead. But something very similar holds for the IVT as well. In accordance with the well-known "epsilon-delta-type" definition of continuity and somewhat surprisingly, the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x^2 < 2 \vee x < 0 \\ -1 & \text{if} \quad x^2 \geq 2 \vee x \geq 0 \end{cases}$$

is continuous on the rational line and yet fails to meet the IVT there. A similar argument holds for other non-Cantorian continua like the Euclidian one (consisting of numbers constructed by means of ruler and compass), the Cartesian one (consisting of algebraic numbers) or that built on *lawlike* sequences of rational numbers, which actually goes back to the Pythagorean definition of proportion by means of a reciprocal subtraction (*anthyphairesis*).[4]

Of course, we have not yet said what a continuum is, but then, neither did

[4] See Fowler (1999).

Bolzano (or Leibniz or Cauchy), so we might as well work merely on the basis of ancient standards where "being continuous" amounts to the simple opposite of "being finitely divisible" and is therefore met already by the totality of rational numbers.

Hence what I say is that since Bolzano did not possess a clear concept of real number, one cannot conclude that he tried to prove – or even proved – some self-evident truth by analytical means. He could not provide a definite "choice" between the truth or falsity of a conjecture since he had no clear concept of truth for arithmetical sentences at all. What we are authorized to claim is merely that Bolzano indicated that for the IVT to hold continuum must be complete in a very specific (order-complete) sense. So instead of a proof we are (at best) facing a decision to define real numbers in a certain holistic way.

In comparison with the IVT, the case of the RT seems to be even trickier since it requires the scrutiny of the concept of definition itself. According to Frege, i.e. from the logician's and logicist's point of view, the usual types of definition, like the definition by recursion, are too specific or unreliable (as Brouwer would say) and should therefore be replaced by, or reduced to, an explicit definition as the only admissible form. For the usual recursive form consisting of two basic steps this means that it has to be expressed by a single formula. The first success of conceptualization or "logification" of arithmetic along these lines was achieved by Frege in his *Begriffsschrift*. He came up with an explicit second-order definition of a closure licensing him to capture natural numbers as the smallest set containing number 0 and closed under a one-to-one successor function *s* which does not assign 0 to any of its arguments. Dedekind achieved the same result putting it in the more familiar form of the so-called Peano's axioms (PA2):

(1) $(\forall x,y)(s(x)=s(y)\rightarrow x=y)$
(2) $(\forall x)(0\neq s(x))$
(3) $(\forall F)(F(0)\wedge(\forall x)(F(x)\rightarrow Fs(x))\rightarrow(\forall y)F(y))$

Just notice that because of the third axiom being second-order this version of arithmetic, contrary to its first order variant (PA1), can actually be conceived as a single axiom.

2

This initial success encouraged Frege to claim that logicism is a feasible *hypothesis* according to which we can expect that (a) numbers will be conceptually separated by means of a single predicate from the domain of all objects or at least from the domain of all "logical objects" (whatever they may be), (b) arithmetical functions will be delimited in a similar way, whereas their "intuitive"

– recursive – formation will first have to be shown to be logistically admissible. This is the task for the RT, as, by the way, Frege's polemic with Grassmann in his *Grundlagen* clearly indicates.

Indeed, thanks to the RT we can dispense with the usual four axioms for addition and multiplication as used in PA1 and, to the same effect, we can introduce the basic arithmetical operations *via* explicit definition

$$x + y = z \quad \text{iff} \quad (\forall f)((\forall x)(f(x,0)=x) \wedge (\forall x,y)(f(x,s(y))=s(f(x,y)))\rightarrow f(x,y)=z),$$

in the same way Frege introduced the concept of natural number

$$Z(y) \quad \text{iff} \quad (\forall F)(F(0)\wedge(\forall x)(F(x)\rightarrow Fs(x))\rightarrow F(y)).$$

All of this comprises (1) the *expressive*, *semantic* part of Frege's logicist project. At the very beginning Frege supplemented it with (2) a *deductive, inferential* pendant, according to which: All arithmetical propositions are to be derived from logical axioms, the conceptual truths of Frege's new logic, by logical rules alone. One of these axioms, the so called G*rundgesetz V* – Frege's axiom of extensionality – provides the ontological basis from which the numbers are to be taken out. This ties both parts of the logicist project – the deductive and the expressive one – together. Let me inspect more closely how and why.

It is not difficult to see that conceptual separation, the process of picking out some objects as falling under a given concept while neglecting others, necessarily presupposes *two* ingredients: (a) the separating concept on the one hand, and (b) the matter or domain from which the objects are to be separated on the other. I call them the (a) *descriptive* and (b) *ontological* ingredients of the semantic part respectively.

It is obvious that the ontological basis we are looking for must be described by logistically acceptable means. The explicit definition, however, cannot be included since it presupposes its *definiens* as already given. What are the options now? Frege chooses to enlarge the logical vocabulary by introducing a second-order term-forming operator {x:Fx} the meaning of which he wishes to fix contextually through *Grundgesetz V* (GV):

$$\{x:Fx\}=\{x:Gx\} \leftrightarrow (\forall x)(Fx\leftrightarrow Gx).$$

In Frege's opinion, this stipulation should supply us with objects utterly independent of non-logical, descriptive vocabulary, or *logical objects*, as he calls them. He does not say it explicitly but the implicit practice in his *Grundgesetze* shows that these are the well-known *pure sets* like the empty set {x:x≠x}, the singleton of the empty set {x:x={x:x≠x}}, and the like.

In this sense, both Frege and Cantor proposed a set-theoretical solution of the foundational problems of arithmetic. But Frege, *unlike* Cantor, realized that now he has to face a new problem. Instead of

“What are numbers and how are they given to us?”,

which was the key issue of his *Grundlagen*, the key question of his *Grundgesetze* is:

“What are sets and how are they given to us?”.

Frege’s general answer, however, is always the same: Numbers, sets or objects in general are something one may be acquainted with only within the context of a proposition. Unfortunately, in the case of sets this proposition – the GV – turned out to be contradictory, bringing Russell’s paradox in its train.

The one thing we must grant to the neologicists is the proven fact that, contrary to common wisdom, Russell’s paradox does not constitute a serious threat to any of the aforementioned parts of the logicist project. GV can be replaced by a similar principle, referred to in the literature as Hume’s Principle (HP),

Card(F)=Card(G) iff F eq G,

whereby two concepts have the same or cardinal number if they are equinumerous (eq), which means that there is a one-to-one correspondence between their extensions. Moreover, Boolos and others showed with the help of very simple analytical model that unlike GV this new principle is deductively consistent with the underlying logic.

As a consequence, HP is inferentially weaker than GV, yet, as already Frege has shown, it is strong enough to entail all the axioms of PA2. That is why this result is known as *Frege’s Theorem* and the L2 together with HP as *Frege Arithmetic*, both due to Boolos. Moreover, by means of RT one can prove the categoricity of the systems in question, as Dedekind explicitly and Frege implicitly did. The semantical completeness of these systems easily follows, which means that Frege and Peano Arithmetic entail every true arithmetical formula and nothing else.

Despite the fact that the concepts of derivation and entailment were by no means clearly established at Frege’s or Dedekind’s time, these results seem to make the idea of logicism being vindicated quite plausible. In the remainder of my paper I shall show why it is plausible only superficially.

3

The first reason is that the whole story has a dramatic sequel, namely the appearance of incompleteness phenomena. According to Gödel's results, not only is arithmetic incomplete in the axiomatic-deductive sense, but the logic it should be prospectively reduced to is deductively incomplete too. The reasoning goes as follows: As we already remarked, PA2 consists of a single axiom, hence if A is an arithmetical formula, then so is the implication with A in the consequent and this axiom in the antecedent. But PA2 is categorical, so if A is the truth of arithmetic, this implication becomes the truth of logic and *vice versa*. From this the incompleteness of the underlying logic easily follows, or to be accurate: it follows that it cannot be weakly complete. (The strong incompleteness is cheaper.)

The fact that we usually do not phrase Gödel's result in this way is motivated by a tacit effort to keep the logic-in-question at least semi-decidable, which is also the real reason lying behind the first- and higher-order distinction and the current paradigm of first-order theories. But since there is no theoretical reason for granting semi-decidability any special status with respect to the logicist project one has to conclude that Gödel's discovery ruined, in effect, its inferential part. As far as the expressive part of the project is concerned our reasoning must be more subtle.

Let us deal with its ontological ingredient first. What we need is an infinite stock of objects serving as the basis for the subsequent conceptual separation. Moreover, this basis should be given in a way independent of any recursive formation. Neologicists try to achieve this by an indirect route adopting Hilbert's standpoint: we cannot say what "point", "line" or "number" are, but only describe their structural properties by means of an axiomatic system. This is the idea of the so-called implicit definition, which, in fact, is already present in Dedekind's logicist account.

Instead of separating numbers from the domain of all objects or logical objects by some *predicate* (like Frege) Dedekind attempted to separate them from the realm of all domains as the unique domain satisfying some *formula*, assuming that there is no need and in fact no way of describing all these possible domains in advance. Unlike Hilbert, Dedekind was fully aware of the fact that his axioms cannot "define" a unique system by themselves, since by definition the set of formulae has always plenty of systems fulfilling them, on condition that they have at least one! But showing that there is such a system that fulfils the axioms of PA2 reduces, as Dedekind quickly realized, to showing again that there is some infinite domain of objects. Hence, we are going round in circles.

To sum up this part: According to the neologicists' view HP seems to provide an infinite domain independent of any recursive formation, but we already know that the reverse is the case. Their HP does not entail but rather presupposes the existence of an infinite domain in order to be consistent.

It is hardly an accident that Boolos' proof of the consistency of HP builds on the model-theoretic-construction consisting of natural numbers, because obviously (1) *natural numbers* constitute the most prominent prototype of an infinite set (2) built up by a simple *recursive process*. The cumbersome examples of an infinite set due to Bolzano and Dedekind:

> proposition A,
> proposition that A is true,
> proposition that the proposition that A is true is true,
> etc.

as much as Frege's, Zermelo's and von Neumann's definition of cardinal and ordinal numbers

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \text{etc.}$$
$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \text{etc.}$$

show very graphically that there is no way of circumventing this direct recursive construction by inventing a larger domain of which they would be species because this domain has to be specified recursively too. In the light of this, we can easily accept Kant's view that what the word "infinity" stands for is only a form of recursion.

Having accepted that the ontological part of the project is not feasible, we can still hope that the descriptive part remains sound; i.e. that provided somebody gives us the individual numbers, we are able to capture them by means of Frege's number-predicate

$$(\forall F)(F(0)\wedge(\forall x)(F(x)\rightarrow Fs(x))\rightarrow F(y)).$$

as a single whole, i.e. as the set to which all and only successors of 0 belong. This is actually no trifling matter since it cannot be done *via* a first order formula or even formulae. Taking a closer look at the arguments for the soundness of the aforementioned number-predicate we find out that they stand or fall with the supposition that its second-order variable ranges over *arbitrary* subsets of the underlying universe including the set of all natural numbers. Hence, even having left aside the problem of impredicativity, we are facing a clear vicious circle, because the set of natural numbers is something we wanted to capture.

To put this last argument into perspective, let us take this infinite formula

$$x=1\vee x=2\vee \text{ etc.}$$

or, in the general case of an arbitrary number set, a formula like this

$$x=1 \vee x=45 \vee x=89 \vee \text{etc.}$$

As far as the intended use is concerned they are sound enough, i.e., they capture the respective sets adequately. But of course they are not formulas in the same sense that second-order predicates (open formulas) are, because they are not finite sequences of characters, and this was actually the only reason Frege employed the second-order expressions in his project.

But what our aforementioned argument pointed at was that the finiteness of these expressions is only apparent because, if they are to work correctly, their variables are to be interpreted as appealing to an arbitrary set. But what else is the arbitrary set but an infinite sequence of numbers or an infinite formula without a generating rule (the meaning of "and so on") which one can actually follow and which therefore must be finite. Hence, from the logicist's and even logician's point of view both – infinite and second-order – predicates must be equally (un)acceptable as long as the aim of writing everything down is to avoid an appeal to somebody's intuition as to what an arbitrary set is, or to what the words "and so on" can mean without knowing "how on".

4

So much for the argument, now comes the conclusion. It was the very decision to define arithmetical objects in the exclusively explicit way which, from the very outset, doomed Frege's foundational program to failure even more decisively than Russell's paradox could. This is because the paradox affects only the project's formal part (which later turned out to be reparable), whereas the logicists (and the so-called neologicists as well) are inevitably forced to employ recursive formations, and not only within the basic formula- or proof-building operations, but also within the justifications of their (second-order) explicit definitions (like that of a closure).

To sum up: Frege's attempt to state or prove something like RT amounts to a decision to perform arithmetic in a certain, very abstract way. In this project, the recursive formations are not conceived as *names* of arithmetical objects, but as their *definite descriptions*, which ought to be checked additionally as for their ability to represent uniquely. This plan turned out to be infeasible, at least in its entirety: a recursion is apparently the simplest way of to constitute or *name* things in arithmetic. It seems probable that Frege eventually realized this and therefore gave up the whole project.

The mainstream of the subsequent foundational movement (which today's neologicism attempts to develop) is a lame compromise between the formalism of the first-order syntax and the Platonism of model or set-theoretical semantics, which does not meet the ambitions of Frege's original plan. Ironically enough, it was the arch-enemy of the verbalized mathematics, the intuitionism of Brouwer, which alone picked up the baton of Frege's basic approach to mathematics, resurrecting as meaningful and non-trivial the seemingly straightforward prescientific questions like "What does it mean for an arithmetical sentence to be true?" and "What are the natural numbers (the so-called standard model of formalized arithmetic) for?".

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