Three

Gödel's Theorems and the Synthetic-Analytic Distinction

Vojtěch Kolman

When posing the old question 'What are arithmetical truths about?' ('What is their epistemic status?' or 'How are they possible?') we find ourselves standing in the shadow of Gödel, just as our predecessors stood in the shadow of Kant. Of course, this observation may be a bit misleading if only for the reason that Gödel's famous incompleteness theorems are not of a philosophical nature, at least not in the first place. There are plenty of texts, however, explaining them as philosophically relevant, i.e. as having some philosophical implications.

In this article I am not aiming to add a new interpretation to the old ones. Rather, I am proposing to see the incompleteness as a link in the chain of certain great (positive or negative) foundational results such as Frege's calculization of logic, Russell's paradox, Gödel's completeness theorem, Gentzen's proof of consistency etc. The foundational line described in this way can then be critically examined as relatively successful with respect to some of its leading ideas and as unsuccessful with respect to others. What I have particularly in mind here is the idea of *reducing*

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arithmetic to logic, with its decisive influence on the rebirth and subsequent development of modern (mathematical) logic. Hence, the key issue of this article may be formulated as follows: 'What do Gödel's theorems tell us about the alleged analyticity or syntheticity of arithmetic?'.

1 Frege's thesis

In the 19^{th} century there were many programs announcing the need of reducing arithmetic to logic (Jevons, Schröder, Dedekind, Peano), yet ironically there was no logic capable of competing with arithmetic in rigor and self-sufficiency. Frege overcame this difficulty by simply inventing it, but his new logic remained virtually unknown until his death, so there must be another explanation of this sudden enthusiasm for logical methods, and, in fact, there is: a general disappointment with the Kantian intuitive conception of mathematics, i.e. with Kant's attempt to ground both arithmetic and geometry in the spatio-temporal structures of reality.

As much as Frege's contemporaries and Frege himself disagreed with Kant on the nature and sources of mathematical knowledge, they were unable to put this disagreement in other than Kantian terms, discarding one side (the left one) of his fundamental distinction between

constructive	vs.	discursive,
intuition	vs.	concept,
mathematic	vs.	logic,
synthetic	vs.	analytic

while endorsing the other. In the light of this observation we can rephrase the main task of Frege's logicism as follows: 'How can one present arithmetic in a non-constructive way?' or 'How can one show that arithmetic is not synthetic, but analytic (not *world*-dependent, but *word*-dependent)?' Or more specifically: 'How can one avoid intuition in the process of justifying basic arithmetical concepts (the intuitive number construction 0, $0+1$, $0+1+1$, etc.) and propositions ($2+2=4$, $34\times 2=68$)?'

The general answer: 'by conceptual means' was, for the first time, successfully implemented in Frege's *Begriffsschrift*, in particular in his ancestral definition and his axiomatization of logic. Proceeding from the (parental) relation R to the explicit second-order definition of its arbitrary finite iteration $RR\ldots R$ (ancestral), Frege was able to rephrase the predicate "x is a number" as "x is a successor of the number 0" or "$(\forall X)[X(0)\wedge(\forall y)(X(y)\rightarrow X(y+1))\rightarrow X(x)]$", with "$x+1=y$" as the basic (parental) relation. This initial success encouraged Frege to declare logicism a feasible *hypothesis*, according to which, prospectively,

(1) numbers are to be *conceptually* separated by means of the aforementioned predicate as a certain *species* of the more general *genus* 'logical object' or of the most general concept 'object' (the same holds for functions whose 'intuitive' recursive — formation was proved to be logistically admissible in Dedekind's famous recursive theorem),[1)]

(2) arithmetical proposition are to be deduced from logical axioms, the conceptual truths of Frege's new logic, by logical rules alone. Among these axioms, the so-called *Grundgesetz V* — Frege's Axiom of Extensionality — is in charge of the ontological basis from which numbers as logical objects are to be separated.

Stages (1) and (2) mirror the expressive and deductive parts, respectively, of the logicist project.

It seems to be clear that Russell's paradox decimated in the first place the second, *proof-theoretical* part of the project. That is why set theorists like Georg Cantor did not regard the antinomy as a serious problem and why some modern logicians, like Crispin Wright or George Boolos, still hope to resurrect logicism in a *model-theoretical*, structuralistic way. The second-order logic they are using ('the set theory in sheep's clothing', as Quine put it) is deductively incomplete (thus proof-theoretically unacceptable), but semantically very strong. This seems to be in accord with the expressive part of the original project.

Although I agree with the neologicists that Frege's system is not affected by the paradox as badly as we thought it was, I claim that this does not warrant it as successful according to Frege's own standards. Let me indicate why.

In a sense, both Frege and Cantor proposed a set-theoretical (i.e. a kind of a semantical) solution to the foundational problems of arithmetic: their numbers are the so-called pure sets or what Frege called 'logical objects'. Frege *unlike* Cantor, however, realized that now we have to face up to a new problem. Instead of 'What are numbers and how are they given to us?' we have 'What are sets and how are they given to us?'. I cannot go into details here,[2)] so I will merely claim that in this respect neither Frege nor anyone else could succeed in a desirable way and that

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[1)] Analyzed in detail in Kolman [2007].

[2)] Cf. Kolman [2005].

it is Russell's paradox that indicates why. To put it simply, there are no objects justifiable by logic alone; there are no pure sets, but only sets of concrete objects like cats, atoms or numbers. And make no mistake! Axiomatic set theory does not justify pure sets, but presupposes them in order to be consistent.

2 Wittgenstein's antithesis

Does all of this imply that the idea of analytical (language-dependent) arithmetic is definitely dead? As a matter of fact, yes, but I do not want to commit myself to such a strong claim yet. Contrary to Poincaré, Brouwer or Wittgenstein, neither do I want to criticize the logicist thesis as such. I pointed out earlier that Frege (unlike Poincaré, Brouwer and Wittgenstein) at least gave his reasons for treating logicism as a bold, but promising *hypo*thesis and that, accordingly, he did not stick dogmatically to it after it was found wrong. However, in spite of this self-critical approach, Frege was clearly too absorbed in his methods to recognize where and why they failed. In this respect, Wittgenstein was more successful.

Like Poincaré and Brouwer in their destructive approach to logicism, Wittgenstein was not overly specific either, but the main idea of his critique is clear: in order to avoid the spatio-temporal intuition Frege strives to proceed as abstractly as possible, pushing the practical arithmetic aside as scientifically irrelevant. But his abstract second-order definitions such as

$$(\forall X)[X(0) \wedge (\forall y)(X(y) \rightarrow X(y+1)) \rightarrow X(x)]$$

work correctly only under the condition that their second-order variables ('X') range over sets specified in pre-scientific, spatio-temporal or practical fashion. In our example it is

> 'the Set of all and only those objects obtained by iterating the operation $+1$ a finite number of times, with 0 at the beginning'.

Since the soundness of these definitions ought to be established before they are employed, the desirable elimination of the practical — presumably Kantian — element is only imaginary, as Wittgenstein clearly recognized in the context of his later reflections on 'following a rule'. His critique of the logicist foundations, however, was already anticipated in his *Tractatus*-theory of internal properties, relations and operations.

According to Wittgenstein, concepts such as number are not explicitly definable, i.e. graspable by means of an explicit formula. Rather, they are categorical notions, and as such they are describable only by means of a rule, i.e. implicitly as a potentially infinite process generating their instances. (This is Wittgenstein's early doctrine of inexpressibility and at the same time Kant's doctrine of concepts as rules.) Wittgenstein represents these internal notions with the help of a complex variable

$$[a, x, O(x)]$$

which can be transcribed in Lorenzen's operativist style as

$$\Rightarrow a, \qquad \text{(starting rule)}$$
$$x \Rightarrow O(x). \qquad \text{(inductive rule)}$$

In the case of number we get the rules '$\Rightarrow |$', '$x \Rightarrow x|$'. Although without any direct *factual* difference (on the arithmetical object level), these transcriptions are very important *philosophically* (on the metalevel). They provide the missing link between the abstract notions of arithmetic and their practical applicability in everyday life. Grasping the concept of number now amounts to mastering the 'counting' rules mentioned above. Wittgenstein, however, did not systematically develop these suggestions into a full-fledged theory since he simply did not believe in such a thing as the 'foundations of arithmetic'. In a sense he may have been right but generally he committed the same error as Frege did, only the other way around: instead of the practice he undervalued the theory.

3 Hilbert's synthesis

The need for the mutual support between practice and theory was already partially recognized by the later Hilbert in his metamathematics. As a study of derivability in certain calculi it was not only a new *mathematical* discipline but also a new *philosophy* of mathematics with the ambition to supersede the old versions of formalism and possibly Brouwer's mentalistically misconceived constructivism. In Lorenzen's later 'operativist' elaboration of Hilbert's ideas[3)] we find certain calculi (collections of rules) as the basic form of mathematical practice (ability to operate according to the rules) and a theory of this operating as 'arithmetic' itself. So, for instance, to say that the arithmetical formula "$2 + 2 = 4$" is true, according to Lorenzen, is to assert the derivability of the figure "$|| + || = ||||$" in the following calculus (+):

$$\Rightarrow x + | = x|, \qquad (+1)$$

[3)] See Lorenzen [1955].

$$x + y = z \Rightarrow x + y| = z|. \qquad (+2)$$

Here the variables x, y, z range over the figures manufactured by this calculus $(|)$:

$$\Rightarrow |, \qquad (|1)$$

$$x \Rightarrow x|. \qquad (|2)$$

This time, however, the variable x is the so-called 'eigenvariable' ranging over the figures so far manufactured by the *same* calculus (the so-called numerals $|$, $||$, $|||$, etc.). As a result, to justify the truth of a sentence such as "$2 + 2 = 4$", one simply has to write down the respective derivation in $(+)$.

So far so good, but the apparent simplicity of this radical syntactic account ends with the next step, namely with the question: 'What is one supposed to write down to justify the falsity of some formula?' Obviously, there is no 'negative' derivation (Russell's negative fact) available. Lorenzen solved this problem by supplementing his operative arithmetic with operative logic, i.e. with something he would have to add later anyway.[4)] His solution is nevertheless very elegant.

According to it (and also according to the approach of the late Wittgenstein), to justify the truth of a sentence does not mean in general just to say the sentence or to write it down, but to justify it to somebody. This somebody needn't be a passive listener, but may potentially disagree, i.e. become an opponent. Constructively interpreted, this implies that the opponent of $\neg A$ commits himself to justifying A. An elementary arithmetical sentence A is true if its proponent can justify it, while its negation $\neg A$ is true if an opponent cannot justify A. Now we can apply the same dialogical approach to the sentential connectives and afterwards build an alternative operational semantics of complex sentences. In this connection Lorenzen uses the name 'dialogical logic'. Instead of giving a systematic account, let me briefly demonstrate the principles it uses by justifying some complex arithmetical sentences. These sentences turn out to be the so-called Peano axioms of arithmetic.

Let us first take the formula $m + 1 = n + 1 \rightarrow m = n$. Translated into the unary notation $m| = n| \rightarrow m = n$, this formula turns out not to depend on the calculus $(+)$, but on another one, which can briefly be described as

$$\Rightarrow | = |, \qquad (=1)$$

4) Lorenzen and Lorenz [1978].

$$x = y \Rightarrow x| = y|. \tag{=2}$$

"Briefly" means that we shall avoid details concerning the general problem of interpreting identity. — By means of semantic tableaux we can now unfold a proponent's justification of the sentence $m| = n| \to m = n$ as a justification of its consequent $m = n$ in the situation of an opponent's simultaneous committing to its antecedent $m| = n|$, where m and n stand for some specific numerals. The proponent's commitment, hence, is conditioned, i.e., he can demand the opponent's reasons before giving his own justification. But this immediately leads to the proponent's victory since every derivation which justifies the antecedent can be converted into a derivation justifying its consequent, simply by deleting the last row. Moreover, since the indicated winning strategy is completely general, i.e. independent of the choice of m, n, we can take our example as a way of justifying the sentence

$$(\forall x, y)(x = y \Rightarrow x| = y|). \tag{P1}$$

The expression "winning strategy" indicates that we are not interested in a victory achieved with the help of good luck but in a victory achieved according to the rules allowing us to win not only against this or that opponent, but against every opponent possible. Otherwise it wouldn't make much sense to call any sentence unambiguously true or false. In the case of a quantified sentence $(\forall x)A(x)$, the relevant rule amounts to a general strategy telling us how to win (a game associated with) the sentence $A(n)$ for a numeral n suggested by a random opponent. Since the sentence

$$(\forall x)(x + 1 \neq x) \tag{P2}$$

doesn't need any additional explanation, we can proceed directly to the induction schema

$$A(|) \wedge (\forall x)(A(x) \to A(x|)) \to (\forall x)A(x). \tag{PI}$$

In order to justify a sentence of the form '$A \wedge B$' one needs to know the winning strategy for both A and B. The beginning of the dialog can be described accordingly as follows:

(O)	(P)	
$A(\vert)$	$(\forall x)A(x)$	(1)
$(\forall x)(A(x) \to A(x\vert))$		(2)

In the next step the opponent chooses some numeral m, thereby attacking the proponent's sentence $(\forall x)A(x)$. The proponent's defence consists simply in writing down the sentence $A(m)$ as a new claim he commits himself to. If we fix some concrete value of m, say $m := \vert\vert\vert$, we can present the further steps schematically in the following tableau, with attacks (?) noted on the right:

$$
\begin{array}{lllll}
 & A(\vert\vert\vert) & (1) & \vert\vert\vert? & (3) \\
A(\vert) \rightarrow A(\vert\vert) & & (2) & \vert? & (4) \\
A(\vert\vert) & A(\vert) & (4) & ? & (5) \\
A(\vert\vert) \rightarrow A(\vert\vert\vert) & & (2) & \vert\vert? & (6) \\
A(\vert\vert\vert) & A(\vert\vert) & (6) & ? & (7)
\end{array}
$$

This strategy is completely general; i.e., the choice of m does not affect the final result of the dialog, but only its length. By iterating steps (4) and (5), the proponent can always force his opponent to claim an elementary sentence $A(m)$ which the proponent was forced to claim before. In this way the tableau 'closes' and we are done.

4 Syntheticity and incompleteness

The relevancy of Lorenzen's operativism to our opening problem is twofold. Firstly, here we have an account of arithmetic which does not start with some undefined or principally indefinable first principles, such as, e.g., Peano axioms, as we have been used to since Hilbert, but tries to justify all the arithmetical propositions (including those 'axioms') by more basic, pragmatic means instead. Being totally in accord with the 'original' characterization of arithmetic as a science dealing with calculations (Kant would add: calculations in time), this account provides us with a *positive* support of the syntheticity thesis. Secondly, we can interpret Gödel's incompleteness result along Lorenzen's line, i.e., as saying that arithmetic does not belong to sciences employing the axiomatic method. This, under appropriate circumstances, can be regarded as support for the syntheticity thesis, too, but in a more *negative* way, which depends partly on the last update we will make to our analyticity/syntheticity distinction and partly on our interpretation of Gödel's theorems. Let us elaborate these two points.

Lorenzen's approach, i.e. his operative arithmetic supplemented with dialogical logic, provides us with a convenient theoretical device for bringing the old distinction between 'analytic' and 'synthetic' or between logic and arithmetic closer to modern standards. First of all, for a given language there is always a certain class of sentences that are justifiable only on the grounds of dialogical rules alone, i.e. their truth (the proponent's winning strategy) doesn't depend on the truth of the elementary sentences in question (that's why it doesn't matter which language is being used). Every sentence of the form '$A \to \neg\neg A$', for instance, belongs to this class of the so-called logical truths.[5)] The arithmetical truth, on the contrary, is defined by means of dialogical rules *plus* arithmetical calculi (|), (=), (+), etc. The winning strategy for (PI), for instance, presupposes the proponent's familiarity with the number-construction via (|). In fact, that's why Lorenzen calls arithmetic 'synthetic' — it is based on construction (*synthesis*). Logical truth, by contrast, is based only on the argumentation-governing rules, i.e. on linguistic norms. As such it may be called 'analytical'.

Instructive as it is, this differentiation doesn't seem to cut deep. Its conventionality points rather in the opposite direction, namely that arithmetic and logic are closely related. They are both dealing with symbols and accordingly may be called 'formal'. In fact, this is Lorenzen's own proposal. His alternative differentiation takes advantage of Gödel's incompleteness result, thereby putting both the technical and the ideological part of modern logic in perspective. The basic idea goes as follows: Just as we have calculized basic arithmetical concepts (such as number, addition, etc.), we can attempt to calculize arithmetical and logical truth. In other words, one may try to describe true arithmetical and logical sentences *only* by means of some mechanical device, manufacturing them as mere syntactic figures according to the respective schematic rules. Knowing that this is *possible* for logic and *impossible* for arithmetic (both due to Gödel), the completeness/incompleteness distinction seems to be finally the required solution to our introductory problem. In the rest of the article, we will discuss this promising possibility, and we will soon discover that things are less easy-going than we might wish. A straightforward presentation of Gödel's incompleteness results turns out to be a necessary part of this enterprise.

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5) Lorenzen's original idea, however, was to show that only the truths of Brouwer's or Heyting's intuitionistic logics are justifiable by the dialogical, i.e. pragmatic means. In the course of the development it turned out that one can dialogically justify both classical and intuitionistic (and in fact many other) concepts, depending on the additional rules which specify (not only *how*, but this time also) *when* it is allowed to attack the opponent and *when* it is allowed to defend oneself against his attack. Hence, speaking of logical truth we need to specify, in advance, what additional dialogical rules we are using.

If we interpret 'analytical' as something like 'free of intuition', thereby indicating that we can possibly relinquish the non-schematic, 'material' controllability of the relevant concepts, then arithmetic post-Gödel is certainly a non-analytical discipline. As we shall emphasize, arithmetical methods of proof *provably* transcend any attempted schematization. However, as we have already pointed out, one cannot conclude from this *negative* evidence that arithmetic *unlike* logic is synthetic, if only for the reason that the original distinction between the constructive and the conceptual is rather blurred in our Hilbertian operative update: they are both dealing with symbols.

My point is that we should first rather examine closer the alleged independence of arithmetic of pure schemata. After all, there are at least two extreme epistemological doctrines of arithmetical truth which consider themselves to be vindicated by the bare fact of incompleteness. The first of them is mathematical mentalism (instantiated in the intuitionism of Brouwer) basing arithmetical truth on mental constructions as opposed to linguistic conventions which, according to Brouwer, are totally heterogeneous with respect to mathematics. The second one is mathematical Platonism with its stress on the independence of arithmetic not only of the individual human subject (which has a point), but of the whole of mankind as well. According to both of them arithmetic is non-analytical. But neither intuitionism nor Platonism, provides us with a satisfactory analysis of how *we* can know that an arithmetical sentence is true and what such a proclamation should mean, not to mention their respective treatments of Gödel's theorems.[6] Let us begin with this.

5 What is an arithmetical rule?

First of all, Gödel's theorems apply only to the theories that are *axiomatized* effectively. This doesn't imply any kind of strong finitism, because we don't want, for example, to rule out axiomatized theories with infinitely many axioms or rules. These axioms and/or rules,[7] however, should be mechanically testable (recognizable), which already implies that they have to be finite sequences of symbols and, as a consequence, that formal deductions (arithmetical proofs) have to be mechanically testable, too. Speaking rather more technically, a theory is effectively

[6] By this I mean especially the most popular versions of their interpretation such as: 'we know that we cannot know everything' (Socratian modesty) or reference to the so-called intuition as a kind of mysterious power available only to the chosen few (the mathematician's 'sixth sense').

[7] Axioms can easily be interpreted as rules and *vice versa*.

axiomatized if and only if it has a *decidable* set of axioms and uses a proof system in which it is *decidable* whether a sequence of well-formed formulas is a proof.

On the other hand, there seems to be no theoretical reason why some infinite rules couldn't under appropriate circumstances be regarded as 'effectively' manageable, too. Let us take, for example, the so-called ω-rule

$$A(|), A(||), A(|||), \text{ etc.} \qquad \Rightarrow \qquad (\forall x)A(x). \tag{ω}$$

As an arithmetical rule it is transparent and sound enough, as long as one interprets the "etc." correctly. In fact, the Gödel-Tarski idea of semantics employs this kind of infinite rules systematically, with the ω-rule as a special case of the more general

$$A(N) \text{ for all substituents } N \qquad \Rightarrow \qquad (\forall x)A(x). \tag{$\forall$}$$

The $\forall$-rule is then nothing but the well-known part of the so-called semantical definition of truth. But let us be careful! Contrary to the ω-rule, the $\forall$-rule doesn't partake in any concrete definition of truth but represents only a truth-schema. The semantical definition ignores the evaluation of elementary sentences.

The whole point of the last paragraph is to make us think about semantic definitions such as ($\forall$) as special (more generously conceived) systems of rules (proof systems) which — starting with some elementary sentences — evaluate the complex ones by *exactly one* of two truth values: true or false. Constructive (effective) or intuitionistic logic denies this very possibility, arguing that from the mere *non-existence* of a winning strategy for $A(x)$ one cannot validly conclude that there is a concrete strategy for some $\neg A(N)$ or, in particular, that the existence of concrete strategies for winning or refuting every $A(N)$ doesn't entail the existence of a general strategy for $A(x)$.

To illustrate my point by a more familiar example, let us consider this: There is no problem in demonstrating whether, for any given even number M, it is the sum of two primes. However, the truth value of the general judgment that every even number is the sum of two primes (the so-called Goldbach conjecture), is still unknown, 250 years after the problem was first posed. Hence, although we could potentially access individual strategies for every single number, we still do not know the general strategy of how to win a proposition concerning them all. Consequently, a decision must be made whether the infinite vehicles of truth and judgment such as (ω) or ($\forall$) should be referred to as rules

(1) only in the case when we positively *know* that all their premises are true, i.e., when we have at our disposal some general strategy for winning all of them at once, or,

(2) more liberally, if we *know* somehow that all their premises are positively true *or* false.

Since the concept of elementary arithmetical truth — as defined by the calculi (=), (+), etc., plus the dialogical rule for negation — is strongly effective, i.e. mechanically testable, we can choose it as our basis. Starting from it we can subsequently arrive at the concept of constructive or classical arithmetical truth, depending on how we interpreted the infinite rule (ω).

Moreover, the classical concept of truth obtained by a more liberal reading of (ω) allows us to articulate the important distinction between the truth we already know and the truth which has not been recognized yet, but is recognizable in principle. (Incidentally, Frege's semantical pair of sense and reference — of truth-conditions and truth-value — aims at the same thing.) What seems to beg the question now is

(1) the compatibility of this mild 'semantical Platonism' (as Stekeler once called it) with the possibility of arithmetical truths which are not only unknown at the moment, but unknowable in principle, on the one hand, and

(2) the widespread opinion that such a strong 'ontological Platonism' is validated by Gödel's theorems, on the other.

In what follows, we can forget about the first part of this question as long as we remember that the problem with Platonism doesn't lie in its compatibility or incompatibility with our experience, but in the lack of better arguments in its favour. The second part, however, is relevant here and we want to refute it in the next section.

6 What is arithmetical truth?

Although the more liberal reading of (ω) gives us a better idea about the so-called standard model of arithmetic, which as we are usually told can be described only in an intuitive way, the constructivists do have a point when saying that the words "rule" or "inference pattern" refer ordinarily to something one can actually follow, hence that a rule which one can follow only *in principle* is not in fact a rule. Imposing the condition of their effective controllability on the premises of (ω) we actually obtain the above mentioned constructive meaning of the quantified sentence $A(x)$: it is true (justified) if and only if there is some general winning strategy for every substitutable name N, i.e. for every sentence $A(N)$.

Since in the constructivist reading the concept of winning strategy remains to a large extent deliberately open, there is always room for an effective, yet liberal enough semantics and a strong effective finite or 'mechanical' syntax or axiomatics. These axiomatics or strong finite rule-systems (i.e. systems with finite rules) are called *full-formalisms*, and those more liberal ones (i.e. systems with infinite rules, no matter if classically or constructively interpreted) are referred to as *semi-formalisms*; both distinctions are due to Schütte [1960]. The important thing is that Gödel's theorems affect only the full-formal systems.

Gödel came up with a general metastrategy of how to construct, for every full-formal (hence schematically given) system of winning arithmetical strategies (i.e. axioms and/or rules), a justifiable arithmetical sentence not winnable by them. This (meta)strategy rests on the so-called diagonal construction and on a presupposition that the starting system is sufficiently strong, since the weak systems are incomplete by definition.

The basic idea behind Gödel's proof looks like this: After devising an appropriate coding scheme we can express many sentences about a certain arithmetical full-formalism in its language and even deduce them in it in accord with the truth. Firstly, we associate arithmetical expressions with numbers (codes) in such a way that a particular number fulfils a concrete arithmetical condition if and only if the encoded expression fulfils a certain syntactic condition, e.g., 'to be an axiom', 'to be a proof' etc. Secondly, we take into consideration the syntactic relation holding between two expressions if and only if the first of them is the proof of the second one and we name the corresponding arithmetical condition

$$\mathrm{Proof}\,(x, y).$$

It holds for two numbers m, n just in case m codes the proof of the formula coded by n. And finally, we need an operation

$$\mathrm{subst}\,(x, y)$$

which yields, for two numbers m, n as arguments, the code p of the result of substituting the numeral for n (i.e. the arithmetical expression 'n') for any occurrence of the sole free variable "x" in the formula $F(x)$ coded by m. Therewith we have the formula

$$(\forall y)\neg\,\mathrm{Proof}\,(y, \mathrm{subst}\,(x, x))$$

at our disposal. Let us abbreviate it as $G(x)$. — So far so good. The formula $G(x)$ has "x" as its sole free variable, it is associated with some code g and it holds of m if the formula coded by $\mathrm{subst}\,(m, m)$ is not provable in the relevant full-formalism, i.e., if there is no number n coding the proof of the formula in question. Substituting the numeral for g in $G(x)$ yields the formula

$$G(\mathrm{g}) \rightleftharpoons (\forall x)\neg\,\mathrm{Proof}\,(x, \mathrm{subst}\,(\mathrm{g}, \mathrm{g})).$$

This is, in fact, the critical formula we are looking for, because $G(\mathrm{g})$ is true (justifiable) if and only if there is no number m such that $\mathrm{Proof}\,(\mathrm{m}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ holds, in other words: if and only if the formula coded by $\mathrm{subst}\,(g, g)$ is unprovable in the given full-formalism. But this formula is $G(\mathrm{g})$ itself! Since the full-formalism is constructed in such a way that it deduces only true (justifiable) arithmetical sentences, the formula $G(\mathrm{g})$ cannot be provable, otherwise we would have a false theorem. Hence, $G(\mathrm{g})$ is not provable, hence $G(\mathrm{g})$ is true!

The reason why we retell the whole story in terms of the semi- and full-formalisms and the winning strategies lies in the observation that the unprovable yet true arithmetical sentence of Gödel's theorem is an unprovable sentence of the full-formalism but a provable (i.e. justifiable or true) sentence of the semi-formalism: there is a strategy of how to win $G(\mathrm{g})$, but also of how to construct a new unprovable yet true formula in the case when $G(\mathrm{g})$ is added *ad hoc* to the original full-formalism as a new axiom. Now it is quite clear how the concept of essential unprovability is *not* to be understood, namely Platonistically, no matter if in a strong or a mild sense. The reason is that Gödel's proof does not even cross the border of the constructive semi-formalism. — Hereafter, therefore, we will carefully differentiate between the deducibility in a full-formalism and in a semi-formalism, using the well-established symbols "$\vdash$" and "$\models$", respectively. To sum up, the theses we have established so far are the following:

(1) The essential incompleteness or incompletability of arithmetic affects only the arithmetical full-formalism, which means that there is always a true arithmetical formula which is not a theorem.

(2) 'Incomplete' thereby always means 'incomplete with respect to some semi-formalism' that defines which sentences are to be evaluated as true and which not.

(3) For this reason the semi-formalism itself cannot be incomplete. However, it can be incomplete with respect to some other semi-formalism, which is actually the case of the constructive semi-formalism in relation to the classical one.

In this particular case, however, we are not able to prove this incompleteness to be essential, so we can keep on believing with Hilbert that every mathematical problem is solvable and we can prospectively announce this general solvability as a kind of regulative (optimistic) hypothesis ("wir müssen wissen, wir werden wissen").[8)]

(4) Even if we interpret arithmetical truth classically, i.e. not effectively, the unprovable yet true formula of Gödel's theorem remains constructively true, i.e. provable in the constructive semi-formalism.

Therefore we should formally differentiate between $\vDash_{\mathcal{C}}$ and $\vDash_{\mathcal{L}}$ ($\mathcal{C}$ standing for "classical", $\mathcal{L}$ for "Lorenzen").

7 Is arithmetic consistent?

Along the same lines we have explained Gödel's theorem we can now handle its corollary, better known under the name of the Second Incompleteness Theorem. This is in fact the famous slogan that the consistency of arithmetic cannot be proved by appealing to arithmetical means alone, or even that it cannot be established at all. Of course, we ought to be cautious here again. A sober reading of the corollary entails only something like this:

> There is an actual strategy of how for every full-formalization T of arithmetic (which is, again, strong enough) to construct a sentence S_T of which the following conditions hold:
>
> (1) S_T is justifiable ($\vDash$) if and only if T is consistent,
>
> (2) S_T is unprovable ($\vdash$) in T if and only if T is consistent.

Then, on closer look, it is clear that the condition (2) alone amounts to a triviality: we can simply take S_T to be any contradiction we like. Adding the condition (1) we want the unprovable sentence of (2) to be true. But

8) [Hilbert, 1935, p. 387].

there is a sentence like this already due to the (First) Theorem. So what is the point of the corollary?

To understand its significance we have to look at it with Hilbert's eyes, i.e. from the point of view of a proponent of strong effective — full-formal — systems. Then there is no place for a semi-formal justification and for a general, non-effective concept of arithmetical truth in particular. According to Hilbert, semantical concepts such as this one could lead us only badly astray (as Russell's paradox had shown) and just for this reason they should eventually be replaced with something more secure, e.g., with syntactical consistency. With this in mind we can first rephrase the concept of incompleteness in syntactic terms and say that an axiomatic (full-formal) theory T is incomplete if there is a sentence S of its language such that neither S nor $\neg S$ are provable ($\vdash$) in T. More importantly, Gödel's theorem can be generalized in this way, i.e. without assuming the truth of the full-formal system in question. The argument goes roughly like this.

Firstly, one must show that the elementary syntactic properties (formulahood, axiomhood, proofhood) are not only adequately expressible in arithmetical language, which means that the arithmetical formulas assigned to these properties are provable or refutable in the arithmetical semi-formalism in accordance with truth, but that these formulas are provable or refutable in the relevant full-formal system T as well. This second characteristic is known as the case-by-case capturing in T and holds of the aforementioned syntactic properties already in the Robinson Arithmetic (Q), i.e. in the (first-order) Peano Arithmetic (PA) without the induction.

Now consider the sentence $G(\mathrm{g}) = (\forall y)\neg\,\mathrm{Proof}\,(y, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ again and suppose it is provable in T. Then there is some number m which codes its proof and, by definition, the sentence $\mathrm{Proof}\,(\mathrm{m}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ is provable in T, too. But the provability of $(\forall y)\neg\,\mathrm{Proof}\,(y, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ entails the provability of $\neg\,\mathrm{Proof}\,(\mathrm{n}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ for every n, particularly for $n := m$. Hence

$$T \vdash \mathrm{Proof}\,(\mathrm{m}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g})) \text{ and } T \vdash \neg\,\mathrm{Proof}\,(\mathrm{m}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g})),$$

which means that T is (syntactically) inconsistent. Hence, if T is consistent, which is the formalist's substitute for truth, there can be no proof of $G(\mathrm{g})$ in T. In order to show that $\neg G(\mathrm{g})$ is unprovable too, we need a stronger assumption than a mere consistency of T. — Gödel introduced the concept of ω-consistency, defining a theory to be ω-consistent if the fact that $T \vdash \neg A(n)$ for each n excludes the possibility of $T \vdash (\exists x)A(x)$.

Note that ω-consistency implies plain consistency, because T is inconsistent if and only if all formulas are provable in it, so in the case of an inconsistent theory the forbidden combination of provable formulas mentioned in the definition of ω-consistency obtains automatically. — Suppose that T is ω-consistent and $\neg G(\mathrm{g})$ is provable in T. Then $G(\mathrm{g})$ is unprovable, because of plain consistency, which means that no number m can code a proof of $G(\mathrm{g})$. Hence, by definition, $\neg\,\mathrm{Proof}\,(\mathrm{n}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$ holds ($\models$) for each n and, by the requirement that T captures the relation 'to be a proof of', even

$$T \vdash \neg\,\mathrm{Proof}\,(\mathrm{n}, \mathrm{subst}\,(\mathrm{g}, \mathrm{g})) \text{ for each } n.$$

But we are assuming that $\neg G(\mathrm{g})$ is provable in T, which is equivalent to $T \vdash (\exists y)\,\mathrm{Proof}\,(y, \mathrm{subst}\,(\mathrm{g}, \mathrm{g}))$, and that makes T ω-inconsistent, contrary to the hypothesis.

Now for the Second Theorem. According to Hilbert, syntactic consistency is a sufficient and in fact the only requirement one can impose on an axiomatic theory. This requirement can be coded in a familiar manner as the unprovability of a contradictory formula in T, i.e. as $(\forall x)\neg\,\mathrm{Proof}\,(x, \mathrm{m})$, where m is, e.g., the code of "$0 = 1$". (This is, of course, because we assume that the negation of "$0 = 1$" is provable in T.) Let us abbreviate $(\forall x)\neg\,\mathrm{Proof}\,(x, \mathrm{m})$ as $\mathrm{Con}\,T$. — The content of the First Theorem in its syntactic version can now be formalized as

$$\mathrm{Con}\,T \rightarrow G(\mathrm{g})$$

and eventually proved in T, if T is at least as strong as PA, hence stronger than Q. Assuming T is consistent, this already yields that $\mathrm{Con}\,T$ must be unprovable, since $T \vdash \mathrm{Con}\,T$ and $T \vdash \mathrm{Con}\,T \rightarrow G(\mathrm{g})$ entail $T \vdash G(\mathrm{g})$, which, as we know from the First Theorem, is not true.

No matter how interesting the unprovability of $\mathrm{Con}\,T$ in T can be, it should be clear that it has nothing to do with the consistency or inconsistency of arithmetic. It makes no sense to construct a formula provable in T if and only if T is consistent since an inconsistent theory entails every formula. In his controversy with Hilbert about the nature of axioms, i.e. of axiomatic theories in general, Frege was right to point out that the consistency of axioms is secondary to their truth: When we are devising an axiomatization of arithmetic, we are obliged, of course, to pick up only true sentences as axioms. Their mutual consistency does not suffice (for them to be picked out), and, as a matter of fact, follows from the fact they are true.

With the proof-theoretical jargon of semi- and full-formalisms at our disposal and with hindsight on the Second Theorem we can now articulate the problem of arithmetical consistency as follows: the already established full-formal systems such as PA or Q are consistent simply because their axioms are provable in the arithmetical semi-formalism. This is, in fact, the usual model-theoretical argument: a theory is consistent because there is a model for it.

The main use of our proof-theoretical diction lies in its relativity. If PA is inconsistent, then arithmetic full-formalism is inconsistent. In the model-theoretical jargon, where semi-formalism is replaced by the so-called standard model, we are usually told that this possibility is precluded simply by definition. Eventually an appeal is made to some kind of intuition. In the proof-theoretical case we do not confine ourselves to such vague justifications, because we can actually prove that the rules of the semi-formalism do not evaluate arithmetical sentences inconsistently. Our method is an easy metainduction:

(1) Elementary arithmetical sentences ($m + n = p$, $m \times n = p$) are evaluated unambiguously as true or false only on the basis of the arithmetical calculi we have set down before,

(2) Tarski's evaluation of complex sentences is correct, too, though we can still argue about whether they assign one and only one value to each sentence (as Stekeler's semantical Platonism maintains)[9] or at most one value to each sentence (as Brouwer and his followers believe).

In fact, this is the difference between the classical and constructivist conception of truth, logic and arithmetic.

8 Conclusion

The import of the Second Theorem seems to lie in its support for the evidence that arguments for the consistency of arithmetic cannot avoid appeal to the infinite, semantical methods. Consistency proofs by Gentzen build on this very idea. In fact, this inevitability is possibly already given in the infinite construction of $|$, $||$, $|||$, $\ldots$, and hence one can, on the one hand, repeat Kant's doubts about arithmetical sentences being deducible from a single formal definition and, on the other hand, expect with Poincaré that this fact has something to do with the complete induction.

[9] See Stekeler-Weithofer [1986].

But we ought to be cautious not to carry this observation too far, as Poincaré [1908] did, having declared complete induction a dividing line between logical and mathematical methods. Of course, complete or mathematical induction is indispensable in mathematics, but this is because numbers (arithmetical operations etc.) are defined inductively, not because it is somehow essential for arithmetic *per se*. Realizing this we can see that induction is indispensable in logic too as long as the basic concepts like formula, theorem or proof are employed. In his *Tractatus* Wittgenstein availed himself of this observation by interpreting natural numbers to be indexes of a sentence-forming operation. Shall we conclude from this evidence that there is no difference between logic and mathematics?

Wittgenstein's later answer would certainly have been negative: Arithmetic and logic are, of course, different, simply by definition: the first one makes computations, the second one inferences. The completeness and incompleteness phenomenon has little to do with this. In fact, instead of interpreting the variable X in the formula

$$(\forall X)[X(0) \wedge (\forall y)(X(y) \to X(y+1)) \to X(x)]$$

schematically, i.e. as ranging over expressions of a formal language, as the first-order version of the induction axiom (schema) does, we can make use of the original, indefinite way, which amounts to the second-order axiom. This is the part of the full-formal system of second-order Peano Arithmetic (PA_2) which is, of course, incomplete due to Gödel's Theorem. We can, however, argue that in contrast with PA the axioms of PA_2 describe arithmetical semi-formalism so well (or technically: up to isomorphism), that every semi-formalism which entails them (under some structure-preserving translation) already entails all theorems of the original semi-formalism (under the same structure-preserving translation). This is nothing but the well-known categoricity theorem for the second-order Peano Arithmetic. As a consequence, to justify an arithmetical sentence we do not need elementary arithmetical calculi any more, we need only Peano's axioms. The point is that for any true arithmetical sentence S the conditional

$$\mathsf{PA}_2 \to S$$

becomes justifiable by logic alone, which means that it is a tautology! Assuming the underlying logic is complete in the sense that all and only the tautologies are deducible, the arithmetic becomes complete, too, which is impossible. Hence, by contraposition, second-order logic is also incomplete. So, Gödel's Theorem, in fact, proves the incompleteness not only for arithmetic, but for logic as well.

But then, we need to ask, what is the moral of Gödel's Incompleteness Theorem with respect to the question of the epistemic status of arithmetic or logic? Are there any essential differences between them? Of course there are, but they are not easily to be found, given the condition that there are remarkable similarities between counting and judging and these undoubtedly are built into the foundations of modern logic. The problem of our question and of the postlogicist philosophy of mathematics in general lies precisely in the fact that they do not take this into account.

So Poincaré was actually right. Modern logic was successful because of availing itself of methods peculiar to mathematics, especially of complete induction. But this did not turn logic into mathematics, as Poincaré suggested; neither did it turn mathematics into logic as the logicists thought. The relations between them were nevertheless changed or distorted, if we wish, and it made them, or certain parts of them, simultaneously more powerful with respect to some problems and objections and more vulnerable with respect to others. But this seems to be a necessary epiphenomenon of any scientific development. — Nevertheless, under the influence of logic, mathematics has become more sensitive to the syntactic design of its theories, making them available for intersubjective checking by devising a transparent, uniform concept of (deductive) proof. The negative side of this move was, of course, the above mentioned identification of arithmetical truth with deductive consistency. Gödel's theorems, as we have interpreted them above, are only the symptoms of some consequences of this decision.

In this article I have tried to argue that they tell us nothing fatal about the nature of our reason, nor anything about logic and arithmetic as its prominent offspring, as long as we are aware that they are (by definition) disciplines of their own, as Wittgenstein used to stress. Simultaneously I want to point out, *pace* the radical scepticism of Poincaré and Wittgenstein, that the story of modern logic shows us how fruitful the possible crossovers of these two disciplines of pure reason can be if they are interpreted in a modest, dialectical way, i.e. not as the reduction of the whole of logic to arithmetic nor *vice versa*, but as the projection of a part of the former onto part of the latter, leading eventually to a discipline of a new, somewhat mixed kind, as already displayed by subjects such as metamathematics, proof- or model-theory, computational complexity and many others.

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